Mathematical Logic : Gödel's First Incompleteness Theorem

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    These works led to the *set theory* of CANTOR (end of the XIX$^{\text{th}}$ century).
  - The beginning of a new field : the mathematical logic.
    This talk is focused on this second point.

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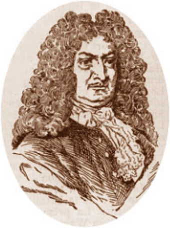

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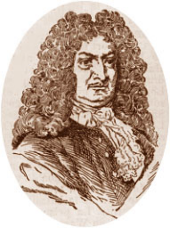

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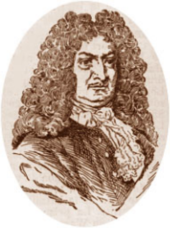

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Symbolic logic was born!

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BOOLE brought these laws to the pair $\{0, 1\}$:

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0	0	1
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You can check he defined an algebra (structure).
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The next year AUGUSTUS DE MORGAN also published two well-known laws in *Formal Logic or The Calculus of Inference* :

$$(1 - xy) = (1 - x) + (1 - y) \text{ and } (1 - (x + y)) = (1 - x)(1 - y)$$

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Notice that some logicians contest classical logic because of its manichaeism induced by the law of excluded middle (e.g. BROUWER).

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This period is called the *foundational crisis*.

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Definition (Formal demonstration)

A formal demonstration is a finite sequence of propositions which starts with a finite number of already demonstrated propositions, including axioms, and where a new proposition is deduced by using inference rules on previous propositions.

Definition (Consistency)

An axiomatic theory is said to be consistent if it doesn't admit a proposition P such as we can demonstrate P and its negation.

Definition (Completeness)

An axiomatic theory is said to be complete if it doesn't admit an undecidable proposition.
And a well-formed proposition P is said undecidable if the theory can't demonstrate either P or its negation.

Take care : true $\neq$ demonstrable

We have "P admits a demonstration $\Rightarrow$ P is true".
But, if the theory isn't complete, the converse isn't right.
If we have an undecidable proposition we have to use arguments out of the theory to determine if it's true or false, they are called metamathematical arguments. So we say (considering P or its negation if P is false) that *true but undemonstrable propositions* exist.

Notation

Instead of writing "P admits a demonstration (in the theory)", we shall write $\vdash P$.

Propositional calculus

We define the propositional calculus, here is the formal language :

Constants		
Propositional connecters		
Symbol	Syntax	Meaning
$\neg$	$\neg p$	Negation, NO-p
$\rightarrow$	$p \rightarrow q$	Imply, if p then q
Punctuation signs		
(		
)		

Propositional variables
$p,\ q,\ p_1,\ p_2,\ p_3 \ldots$

And we define $(p \vee q) \equiv ((\neg p) \rightarrow q)$ (disjunction) and $(p \wedge q) \equiv \neg(p \rightarrow (\neg q))$ (conjunction).

The propositional calculus has only one inference rule, called *modus ponens* :

Modus ponens

If we have $\vdash P$ and $\vdash P \rightarrow Q$ then we have $\vdash Q$.

It has 3 axioms (some authors don't use the same axioms, but get the same result, they demonstrate these axioms and we can demonstrate their axioms) :

Axioms

1. $(p \rightarrow (q \rightarrow p))$
2. $((p_1 \rightarrow (p_2 \rightarrow p_3)) \rightarrow ((p_1 \rightarrow p_2) \rightarrow (p_1 \rightarrow p_3)))$
3. $(((\neg q) \rightarrow (\neg p)) \rightarrow (((\neg q) \rightarrow p) \rightarrow q))$

The 3$^{\text{rd}}$ axiom is just the *reductio ad absurdum*.

I chose these axioms because they allow to easily demonstrate this useful metatheorem :

Theorem (The deduction (meta)theorem)

If $P \vdash Q$ *then* $\vdash (P \to Q)$.
Where $P \vdash Q$ *means that if* P *admits a demonstration then* Q *too.*

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Where $P \vdash Q$ means that if P admits a demonstration then Q too.

We can show that the theory we have just defined is consistent and complete.

KURT GÖDEL
(1906-1978)

In 1931 GÖDEL participated to the rigorous definition of theories introducing *computable functions* and *recursive sets*...
The same year he also demonstrated that a large family of theories are incomplete and that another large family of consistent theories aren't able to demonstrate their consistency themselves.
This talk is focused on the first theorem, called *Gödel's first incompleteness theorem*.

Note that GÖDEL demonstrated this theorem in a particular theory (based on the theory of the *Principia Mathematica* adding PEANO's axioms). But we have now rigorous and general demonstrations, some use TURING machines, and some just a formal system (and are more general).
This talk presents the second one and the goal of such a demonstration is to find out a well-formed proposition which is undecidable.

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The same idea is used to find out an undecidable proposition which means *I'm not demonstrable*.

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But this paradox isn't satisfactory, actually "(not) to be richardian" isn't an arithmetic property. It makes sense only after we ordered the arithmetic properties.

In 1931 GÖDEL succeeded in demonstrating that a theory, obtained by adding Peano's axioms to the Principia Mathematica, is incomplete. He took inspiration from RICHARD's paradox but he found (after a lot of lines of logic demonstration and some hypothesis) a well-formed proposition in the theory which means "I'm not demonstrable". Then he extrapolated (which isn't very rigorous) to a large family of theories.
The theorem was generalized (i.e. demonstrated with more general hypothesis), for example by ROSSER who chose another well-formed sentence. And we now have rigorous demonstrations.

All theories able to describe the whole arithmetic or our currently theory used to describe the set theory are concerned by the theorem and so admit undecidable propositions.
A well known undecidable theorem in set theory (ZFC theory) is the *continuum hypothesis* which is : "the set of real numbers has minimal possible cardinality which is greater than the cardinality of the set of integers".
Formally, it states that a set E verifying $\mathrm{card}\,\mathbb{N} < \mathrm{card}\,E < \mathrm{card}\,\mathbb{R}$ doesn't exist.
But some theories are complete, for example the propositional calculus or the TARSKI's Euclidian geometry theory.

The main effect is that we shall never find a universal axiomatic theory for all the mathematics which is complete and consistent.

Which was one of HILBERT's dreams (1862-1943)...

If you want a rigorous demonstration with rigorous hypothesis you can check `http://citron.9grid.fr/documents.html#doc7`